ON A PROBLEM OCCURRING IN THE THEORY OF OPTIMUM CONTROL WITH AFTEREFFECTS

(OB ODNOI ZADACHE, VOZNIKAIUSHCHEI V TEORII OPTIMAL'NOGO REGULIROVANIIA S POSLEDEISTVIIAMI)

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V.R. NOSOV

(Moscow)

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1. We will consider the following problem of optimum control for systems with aftereffects. Let there be given the system (1.1)

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + M(t)u(t), \quad x(t) = \varphi(t), \quad t \in [a-\tau, a]$$

and the functional

$$I[u] = \frac{1}{2}\int_a^b \{x^*(t)F(t)x(t) + x^*(t-\tau)G(t)x(t-\tau) + u^*(t)H(t)u(t)\}\,dt \qquad (1.2)$$

Here $x(t)$ and $u(t)$ are m-dimensional vectors, A, B, M, F, G, H are quadratic matrices.of the mth order, where F, G, H are positive-definite matrices and $G(t) = 0$ for $t > b$. It will be regarded that all matrix elements are continuous functions of time. The initial vector function $\varphi(t)$ is also regarded continuous in $[a-\tau, a]$.

It is required to determine Equation $u(t)$ so that the functional (1.2) have a minimum. We regard the control as permissible if it is piece-wise differentiable.

As was shown by Halanay [1 and 2] the problem of finding the optimum control for the system (1.1) and the functional (1.2) is equivalent to the following boundary-value problem:

$$\begin{aligned} &x'(t) = A(t)x(t) + B(t)x(t-\tau) - M(t)H^{-1}(t)M^*(t)z(t) \\ &z'(t) = -A^*(t)z(t) - B^*(t+\tau)z(t+\tau) - [F(t) + G(t+\tau)]x(t) \qquad (1.3) \\ &x(t) = \varphi(t), \quad t \in [a-\tau, a]; \qquad z(t) = 0, \quad t \in [b, b+\tau] \end{aligned}$$

If the problem (1.3) has a solution, then

$$u(t) = -H^{-1}(t)M^*(t)z(t)$$

will be the optimum control in the problems (1.1), (1.2).

Let us consider a more general problem than (1.3)

$$\begin{aligned} &x'(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)y(t) \\ &y'(t) = -A^*(t)y(t) - B^*(t+\tau)y(t+\tau) + D(t)x(t) \qquad (1.4) \\ &x(t) = \varphi(t), \quad t \in [a-\tau, a]; \qquad y(t) = \psi(t), \quad t \in [b, b+\tau] \end{aligned}$$

As was shown in [1], the problem (1.4) can be reduced to the Fredholm integral equation of second kind. But the kernel of this equation is not expressible explicitly in terms of the coefficients of the problem (1.4). Such reduction therefore does not yield simple conditions for the existence of a solution for the problem (1.4).

The present paper will establish certain simple sufficient conditions for the existence of a unique solution of the problem (1.4), or in other words, there shall be obtained the conditions for the existence of the optimum control for the problem (1.1), (1.2).

2. Consider the following problem:

$$x'(t) - A(t)x(t) - B(t)x(t-\tau) + C(t)y(t) = f_1(t)$$

$$y'(t) + A^*(t)y(t) + B^*(t+\tau)y(t+\tau) + D(t)x(t) = f_2(t) \tag{2.1}$$

$$x(t) = 0, \quad t \in [a-\tau, a]; \qquad y(t) = 0, \quad t \in [b, b+\tau] \tag{2.2}$$

It will be shown below that the problem (1.4) can easily be reduced to the problem (2.1). The systems of equations (2.1) will be expressed in the form of a single operator equation

$$L\{x(t),\ y(t)\} = \{f_1(t),\ f_2(t)\} \tag{2.3}$$

We will denote by W a Hilbert space of the continuous in $[a, b]$ mth dimensional vector possessing a quadratically integrable first derivative. The norm in this space is defined as follows:

$$\|x^2\|_W = \int_a^b \{x^2(t) + x'^2(t)\}\, dt \tag{2.4}$$

The space W is complete with respect to this norm.

We will denote by D_0 a subspace of the space $W \times W$ the coordinates of which satisfy the condition (2.2), and by L_2 the vector space the coordinates of which are quadratically summable in $[a, b]$.

Let us consider the problem conjugate to the problem (2.1) and (2.2)

$$X'(t) - A^*(t)X(t) - B^*(t+\tau)X(t+\tau) + D^*(t)Y(t) = g_1(t)$$

$$Y'(t) + A(t)Y(t) + B(t)Y(t-\tau) + C^*(t)X(t) = g_2(t) \tag{2.5}$$

$$X(t) = 0, \quad t \in [b, b+\tau]; \qquad Y(t) = 0, \quad t \in [a-\tau, a] \tag{2.6}$$

or in short, (2.5) can be expressed in the form

$$M\{X(t), Y(t)\} = \{g_1(t), g_2(t)\}$$

Here the operator M is conjugate to L. Along with the solutions of the problem (2.5), (2.6), we will investigate the generalized solutions from the space D_0.

The vector $\{X(t), Y(t)\} \in L_2$ will be termed the generalized solution of the problem (2.5),(2.6), if it satisfies the following integral identity:

$$\int_a^b \{X^*(t)[\Phi_1'(t) - A(t)\Phi_1(t) - B(t)\Phi_1(t-\tau) + C(t)\Phi_2(t)] +$$

$$+ Y^*(t)[\Phi_2'(t) + A^*(t)\Phi_2(t) + B(t+\tau)\Phi_2(t+\tau) +$$

$$+ D(t)\Phi_1(t)]\}\, dt = \int_a^b \{g_1^*(t)\Phi_1(t) + g_2^*(t)\Phi_2(t)\}\, dt \tag{2.7}$$

$$\{\Phi_1(t),\ \Phi_2(t)\} \in D_0$$

It is easy to see that each solution of the problem (2.5),(2.6), belonging to the class D_0, satisfies the identity (2.7). For this it is sufficient to multiply Equation (2.5) by $\Phi_1(t)$ and $\Phi_2(t)$, to integrate over $[a, b]$ to take the first integral in parts and to take into account the boundary conditions for $\{X(t), Y(t)\}$ and $\{\Phi_1(t), \Phi_2(t)\}$.

Let us assume now that the matrices $C(t)$ and $D(t)$ are symmetric and positive-definite, i.e. that for all $t \in [a, b]$

$$C(t) = C^*(t), \qquad D(t) = D^*(t) \tag{2.8}$$

$$\xi^* C(t)\xi \geqslant \alpha\xi^*\xi; \qquad \xi^* D(t)\xi \geqslant \beta\xi^*\xi \qquad (\alpha, \beta > 0) \tag{2.9}$$

Here ξ is an arbitrary m-dimensional vector. It will be shown below that for any solutions of the problem (2.1),(2.2) and the problem (2.5),(2.6) the following inequalities are valid:

$$\|x\|_W + \|y\|_W \leqslant C_1 \{\|f_1\|_{L_2} + \|f_2\|_{L_2}\} \tag{2.10}$$

$$\|X\|_{L_2} + \|Y\|_{L_2} \leqslant C_2 \{\|g_1\|_{L_2} + \|g_2\|_{L_2}\} \tag{2.11}$$

It follows from the inequality (2.10) that the problem (2.1),(2.2) can have only one solution. Also [3], it follows from (2.10) that the operator L is a closed operator in the space D_0, or in other words, the set of values $R(L)$ of the operator L is a closed subspace of the space L_2. We will show that $R(L)$ coincides with the entire L_2, i.e. that the operator implements the mutually single valued mapping of the space D_0 on L_2. This means that for any $\{f_1, f_2\} \in L_2$ there shall be such $\{x, y\} \in D_0$, that $L\{x,y\} = \{f_1, f_2\}$, or that the problem (2.1),(2.2) has a solution for all $\{f_1, f_2\} \in L_2$. For the proof of the coincidence of $R(L)$ and L_2 we will show that L_2 has no element of $\{x,y\}$ different from zero and orthogonal to all $R(L)$, i.e. we will show that if

$$\int_a^b \{X^*(t)[x'(t) - A(t)x(t) - B(t)x(t-\tau) + C(t)y(t)] + \tag{2.12}$$

$$+ Y^*(t)[y'(t) + A^*(t)y(t) + B^*(t+\tau)y(t+\tau) + D(t)x(t)]\}\,dt = 0, \qquad \{x, y\} \in D_0$$

then it follows that $X(t) \equiv 0$, $Y(t) \equiv 0$.

Comparing (2.12) and (2.7) we see that the last statement coincides with the uniqueness theorem for the generalized solution of the problem (2.5), (2.6) but the uniqueness theorem of the generalized solution of the problem (2.5),(2.6) follows from the inequality (2.11). Indeed, the inequality (2.11) indicates that the operator M can be expanded over all space L_2 and that it is closed in L_2. Also, according to a known theorem ([3], p. 555) we find that for any $\{X, Y\} \in L_2$ the inequality

$$\|X\|_{L_2} + \|Y\|_{L_2} \leqslant C_2 \|M^\circ\{X, Y\}\|_{L_2} \tag{2.13}$$

is valid where M° denotes the expansion of the operator M over L_2. We note, finally, that the generalized solution of the problem (2.5),(2.6) can be determined not only with the aid of the identity (2.7) but also as a solution of the operational equation

$$M^\circ\{X, Y\} = \{g_1, g_2\} \tag{2.14}$$

But the solution of Equation (2.14), or the generalized solution of the problem (2.5),(2.6), satisfies the inequality (2.11). The uniqueness theorem of the generalized solution for the problem (2.5),(2.6) follows directly from the inequality (2.11). Thus, if the inequalities (2.10),(2.11) are satisfied then it is proved that the problem (2.1),(2.2) is uniquely solvable for any $\{f_1, f_2\} \in L_2$. The functions from D_0 are continuous in $[a,b]$ and

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) - C(t)y(t) + f_1(t)$$

Consequently, any solution of the problem (2.1),(2.2) belonging to D_0 has continuous first derivatives if the matrices A, B, C, D and f_1, f_2 are continuous in $[a,b]$.

Thus, the existence proof for the unique solution of the problem (2.1), (2.2) is reduced to the proof of the inequalities (2.10) and (2.11).

We will establish these inequalities. We will prove at first the simpler inequality

$$\|x\|_{L_2} + \|y\|_{L_2} \leqslant C_3 \{\|f_1\|_{L_2} + \|f_2\|_{L_2}\} \tag{2.15}$$

The first equation in (2.1) is dot multiplied by $y(t)$, the second by $x(t)$ and the two equations are summed. Then

$$\frac{d}{dt}(x(t), y(t)) + (C(t)y(t), y(t)) + (D(t)x(t), x(t)) +$$

$$+ (B^*(t+\tau)y(t+\tau), x(t)) - (B(t)x(t-\tau), y(t)) = (f_1(t), y(t)) + (f_2(t), x(t)) \tag{2.16}$$

We integrate now (2.16) over t from a to b. Then on the strength of the conditions (2.2), the first term of this equality vanishes and

$$\int_a^b (B^*(t+\tau)y(t+\tau),\ x(t))\,dt = \int_{a-\tau}^{b-\tau} (B^*(t+\tau)y(t+\tau),\ x(t))\,dt =$$
$$= \int_a^b (B^*(\xi)y(\xi),\ x(\xi-\tau))\,d\xi = \int_a^b (B(t)x(t-\tau),\ y(t))\,dt \tag{2.17}$$

Taking into account (2.17) we find that

$$\int_a^b \{(C(t)y(t),\ y(t)) + (D(t)x(t),\ x(t))\}\,dt = \int_a^b \{(f_1(t),\ y(t)) + (f_2(t),\ x(t))\}\,dt \tag{2.18}$$

From (2.18) and due to (2.8) and (2.9), we find easily that

$$\alpha\|x\|_{L_2}{}^2 + \beta\|y\|_{L_2}{}^2 \leqslant \frac{2}{\beta}\|f_1\|_{L_2}{}^2 + \frac{2}{\alpha}\|f_2\|_{L_2}{}^2 + \frac{\alpha}{2}\|x\|_{L_2}{}^2 + \frac{\beta}{2}\|y\|_{L_2}{}^2 \tag{2.19}$$

Equation (2.15) follows immediately from (2.19). We note that the constant C_3 in the inequality (2.15) is determined only by the quantities α and β.

We establish now (2.10). For this the first equation in (2.1) is dot multiplied by $x'(t)$, the second by $y'(t)$, the equations are integrated in the interval $[a,b]$ and are added. Then to each of the derived integrals we apply the inequality

$$\int_a^b (\xi(t),\ \eta(t))\,dt \leqslant \varepsilon\|\xi(t)\|_{L_2}{}^2 + \frac{1}{\varepsilon}\|\eta(t)\|_{L_2}{}^2$$

and set $\varepsilon = 1/_8$. Then

$$\|x'\|_{L_2}{}^2 + \|y'\|_{L_2}{}^2 \leqslant 8\|A(t)x(t)\|_{L_2}{}^2 + 8\|C(t)y(t)\|_{L_2}{}^2 + 8\|B(t)x(t-\tau)\|_{L_2}{}^2 +$$
$$+ 8\|D(t)x(t)\|_{L_2}{}^2 + 8\|A^*(t)y(t)\|_{L_2}{}^2 + 8\|B^*(t+\tau)y(t+\tau)\|_{L_2}{}^2 + 8\|f_1(t)\|_{L_2}{}^2 +$$
$$+ 8\|f_2(t)\|_{L_2}{}^2 + 1/_2\|x'\|_{L_2}{}^2 + 1/_2\|y'\|_{L_2}{}^2 \tag{2.20}$$

Let the elements of the matrices A, B, C, D be measurable and bounded functions in $[a,b]$. Then utilizing the inequality (2.15) we obtain from (2.20) the inequality (2.10) in which the constant C_1 depends only on the coefficients of the system (2.1).

This proves the inequality (2.10). The inequality (2.11) is established analogously to (2.15). Thus the theorem is fully proved.

T h e o r e m 2.1. Let the matrices A, B, C, D be measurable and bounded in $[a, b]$ and the matrices C and D satisfy the conditions (2.8), (2.9) almost everywhere in $[a, b]$. Then for any $\{f_1, f_2\} \in L_2$ there exists a unique solution of the problem (2.1), (2.2) belonging to the space D_0.

If, in addition, the coefficient matrices of the system (2.1) and the right-hand sides of f_1 and f_2 are continuous in $[a, b]$, then the solution of the problem (2:1), (2.2) has continuous derivatives.

N o t e 1. Let us reduce the problem (1.4) to the problem (2.1),(2.2). It will be shown first that the problem (1.4) can have only a unique solution. Indeed, let $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be two solutions of the problem (1.4). Then $\{x_1 - x_2,\ y_1 - y_2\}$ is a solution of the problem (1.4) with zero boundary conditions. But then it follows from the inequality (2.10) that

$$x_1 - x_2 \equiv 0, \qquad y_1 - y_2 \equiv 0$$

Let us now select the functions $x_0(t)$ and $y_0(t)$ such that

$$x_0(t) = \varphi(t), \quad t \leqslant a; \qquad x_0(t) = \varphi(a), \quad a \leqslant t \leqslant b$$
$$y_0(t) = \psi(t), \quad b \leqslant t; \qquad y_0(t) = \psi(b), \quad a \leqslant t \leqslant b \tag{2.21}$$

Consider the difference $x(t) - x_0(t) = x_3(t)$, $y(t) - y_0(t) = y_3(t)$. It is easy to see that $\{x_3(t), y_3(t)\}$ is a solution of the problem (2.1), (2.2) in which

$$f_1(t) = x_0'(t) - A(t)x_0(t) - B(t)x_0(t-\tau) + C(t)y_0(t)$$
$$f_2(t) = y_0'(t) + A^*(t)y_0(t) + B^*(t+\tau)y_0(t+\tau) + D(t)x_0(t)$$

If φ and ψ are quadratically integrable over their regions of definition, then $\{f_1, f_2\} \in L_2$, and if in addition φ and ψ are continuous then also f_1 and f_2 are continuous.

Having utilized the uniqueness of the solution for the problem (1.4) we see that the assertion of the Theorem 2.1 is valid also for the problem (1.4).

N o t e 2 . Theorem 1 remains valid also when one of the numbers a, b or both simultaneously become infinite. This can be easily seen if it is taken into account that the constants in the evaluations of (2.10),(2.11) are independent of the integration interval, and that the space D_0 is understood as the function space quadratically summable and possessing a quadratically summable first derivative.

N o t e 3 . For the case when the lag depends on time it is possible to prove a type of the Theorem 2.1 for the problem of the form

$$x'(t) - A(t)x(t) - B(t)x(t - \tau(\gamma(t))) + C(t)y(t) = f_1(t)$$
$$y'(t) + A^*(t)y^*(t) + B^*(t + \tau(t))(1 + \tau'(t))y(t + \tau(t)) + D(t)x(t) = f_2(t)$$
$$x(t) = 0, \quad t \leqslant a; \qquad y(t) = 0, \quad t \geqslant b$$

Here $\gamma(t)$ denotes an inverse function to $t - \tau(t)$ (it is assumed that such a function exists). The special form of the equation ensures the fulfillment of (2.17) in this case as well. The remaining proof is literally repeated.

3. We establish still another indication for the existence of a unique solution of the problem (1.4). Let us investigate a more general problem

$$x'(t) = A(t)x(t) + B(t)x(t - \tau(t)) + C(t)y(t) + f_1(t)$$
$$y'(t) = D(t)y(t) + E(t)y(t + \tau(t)) + F(t)x(t) + f_2(t) \tag{3.1}$$
$$x(t) = \varphi(t), \quad t \leqslant a; \qquad y(t) = \psi(t), \quad t \geqslant b \tag{3.2}$$

Here A, B, C, D, E, F are arbitrary continuous quadratic matrices of order m; $\varphi(t)$, $\psi(t)$ and $f_1(t)$, $f_2(t)$ are continuous functions.

The problem (3.1),(3.2) can be reduced to the Fredholm integral equation of second kind, analogously to what was done in [1], and then prove the existence of the solution for problem (3.1),(3.2) by the method of successive approximations. However, since the kernel of this integral equation is not known explicitly, it is more convenient to apply the method of successive approximations directly to the problem (3.1),(3.2).

We will prove the following theorem.

T h e o r e m 3.1 . Let the matrices A, B, C, D, E, F be continuous; φ, ψ, f_1, f_2 are also continuous. Let in addition

$$\int_a^b (\|A(t)\| + \|B(t)\| + \|F(t)\|)\,dt \leqslant \delta < 1 \quad \int_a^b (\|C(t)\| + \|E(t)\| + \|L(t)\|)\,dt \leqslant \delta < 1 \tag{3.3}$$

Then the problem (3.1),(3.2) has a unique solution. Let us pass from the problem (3.1),(3.2) to the integral equations

$$x(t)=\varphi(a)+\int_a^t\{A(\xi)x(\xi)+B(\xi)x(\xi-\tau(\xi))+C(\xi)y(\xi)+f_1(\xi)\}d\xi$$
$$y(t)=\psi(b)+\int_t^b\{D(\xi)y(\xi)+E(\xi)y(\xi+\tau(\xi))+F(\xi)x(\xi)+f_2(\xi)\}d\xi \tag{3.4}$$

Here we take

$$x(t)=\varphi(t),\quad t\leqslant a;\qquad y(t)=\psi(t),\quad t\geqslant b$$

Define $x_0(t)$, $y_0(t)$ the same way as in (2.21) and construct $x_1(t)$ and $y_1(t)$ in accordance with Formulas

$$x_1(t)=\varphi(t)\qquad (t\leqslant a)$$
$$x_1(t)=\varphi(a)+\int_a^t\{A(\xi)x_0(\xi)+B(\xi)x_0(\xi-\tau(\xi))+C(\xi)y_0(\xi)+f_1(\xi)\}d\xi\qquad (t\geqslant a)$$
$$y_1(t)=\psi(t)\qquad (b\leqslant t)$$
$$y_1(t)=\psi(b)+\int_t^b\{D(\xi)y_0(\xi)+E(\xi)y_0(\xi+\tau(\xi))+F(\xi)x_0(\xi)+f_2(\xi)\}d\xi\qquad (t\leqslant b)$$

Then the $x_n(t)$, $y(t)$ are constructed analogously. It is easy to see that in fulfilling the conditions (3.3)

$$\|x_n-x_{n-1}\|_C+\|y_n-y_{n-1}\|_C\leqslant\delta\|x_{n-1}-x_{n-2}\|_C+\delta\|y_{n-1}-y_{n-2}\|_C$$

i.e. that the integral operator (3.4) is the contraction operator in the space $C[a,b]$. It has a single fixed point which is the solution of the problem (3.1),(3.2).

The Theorem 3.1 is proved.

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